Proofs Using Coordinate Geometry

What You’ll Learn
• To prove theorems using figures in the coordinate plane

... And Why
To use coordinate geometry to prove that a flag design includes a rhombus, as in Example 2

In Lesson 5-1, you learned about midsegments of triangles. A trapezoid also has a midsegment. The midsegment of a trapezoid is the segment that joins the midpoints of the nonparallel opposite sides. It has two unique properties.

Formulas for slope, midpoint, and distance are used in a proof of Theorem 6-18.

Planning a Coordinate Geometry Proof
Developing Proof
Plan a coordinate proof of Theorem 6-18.

Given: \( \overline{MN} \) is the midsegment of trapezoid \( \text{TRAP} \).

Prove: \( \overline{MN} \parallel \overline{TP}, \overline{MN} \parallel \overline{RA}, \) and \( MN = \frac{1}{2}(TP + RA) \).

Plan: Place the trapezoid in the coordinate plane with a vertex at the origin and a base along the \( x \)-axis. Since midpoints will be involved, use multiples of 2 to name coordinates. To show lines are parallel, check for equal slopes. To compare lengths, use the Distance Formula.

EXAMPLE
Planning a Coordinate Geometry Proof

### Key Concepts

**Theorem 6-18**

Trapezoid Midsegment Theorem

1. The midsegment of a trapezoid is parallel to the bases.
2. The length of the midsegment of a trapezoid is half the sum of the lengths of the bases.

\[ MN \parallel TP, MN \parallel RA, \] and \[ MN = \frac{1}{2}(TP + RA). \]

Formulas for slope, midpoint, and distance are used in a proof of Theorem 6-18.

LESSON 6-6

**Check Skills You’ll Need**

1. Graph the rhombus with vertices \( A(2, 2), B(7, 2), C(4, -2), \) and \( D(-1, -2) \). Then, connect the midpoints of consecutive sides to form a quadrilateral. What do you notice about the quadrilateral? The quad. is a rectangle.

**Algebra**

2. Give the coordinates of \( B \) without using any new variables.

**Building Proofs in the Coordinate Plane**

In Lesson 5-1, you learned about midsegments of triangles. A trapezoid also has a midsegment. The midsegment of a trapezoid is the segment that joins the midpoints of the nonparallel opposite sides. It has two unique properties.

**New Vocabulary**

• midsegment of a trapezoid
Complete the coordinate proof of Theorem 6-18. See margin, p. 350.

a. Find the coordinates of midpoints M and N. How do the multiples of 2 help?
b. Find and compare the slopes of \( MN \), \( TP \), and \( RA \).
c. Find and compare the lengths \( MN \), \( TP \), and \( RA \).
d. In parts (b) and (c), how does placing a base along the \( x \)-axis help?

Algebra

The rectangular flag at the left is constructed by connecting the midpoints of its sides. Use coordinate geometry to prove that the quadrilateral formed by connecting the midpoints of the sides of a rectangle is a rhombus.

**Given:** \( MNPO \) is a rectangle.

\( T, W, V, U \) are midpoints of its sides.

**Prove:** \( TWVU \) is a rhombus.

**Plan:** Place the rectangle in the coordinate plane with two sides along the axes. Use multiples of 2 to name coordinates. A rhombus is a parallelogram with four congruent sides. From Lesson 6-6, Example 1, you know that \( TWVU \) is a parallelogram. To show \( TW \cong WV \cong VU \cong UT \), use the Distance Formula.

**Coordinate Proof:** By the Midpoint Formula, the coordinates of the midpoints are

\[ T(0, b), W(a, 2b), V(2a, b), \text{ and } U(a, 0) \]

By the Distance Formula,

\[ TW = \sqrt{(a - 0)^2 + (2b - b)^2} = \sqrt{a^2 + b^2} \]

\[ WV = \sqrt{(2a - a)^2 + (b - 2b)^2} = \sqrt{a^2 + b^2} \]

\[ VU = \sqrt{(a - 2a)^2 + (0 - b)^2} = \sqrt{a^2 + b^2} \]

\[ UT = \sqrt{(0 - a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2} \]

\( TW \cong WV \cong VU \cong UT \), so parallelogram \( TWVU \) is a rhombus.

**Critical Thinking**

Explain why the proof using \( M(0, 2b), N(2a, 2b), P(2a, 0), \) and \( O(0, 0) \) is easier than a proof using \( M(0, b), N(a, b), P(a, 0), \) and \( O(0, 0) \).

See back of book.

**EXERCISES**

**Practice and Problem Solving**

For more exercises, see Extra Skill, Word Problem, and Proof Practice.

**Practice by Example**

**Example 1** (page 348)

1. \( W \) and \( Z \) are the midpoints of \( OR \) and \( ST \), respectively. In parts (a)–(c), find the coordinates of \( W \) and \( Z \). 
   a. \( W \left( \frac{a}{2}, b \right) \); \( Z \left( \frac{c + e}{2}, d \right) \)
   b. \( W(a, b); Z(c + e, d) \)
   c. \( W(2a, 2b); Z(2c + 2e, 2d) \)
   d. \( c; \) it uses multiples of 2 to name the coordinates of \( W \) and \( Z \).

**GO for Help**

1. a. \( W \left( \frac{a}{2}, \frac{b}{2} \right); Z \left( \frac{c + e}{2}, \frac{d}{2} \right) \)
   b. \( W(a, b); Z(c + e, d) \)
   c. \( W(2a, 2b); Z(2c + 2e, 2d) \)
   d. \( c; \) it uses multiples of 2 to name the coordinates of \( W \) and \( Z \).

2. a. \( y \)-coordinate of midsegment: \( \frac{b}{2} \)
   b. \( y \)-coordinate of midsegment: \( \frac{d}{2} \)
   c. \( y \)-coordinate of midsegment: \( \frac{d}{2} \)
   d. \( y \)-coordinate of midsegment: \( \frac{d}{2} \)

**Advanced Learners**

After reading Example 2, students should be able to prove that the quadrilateral formed by connecting the midpoints of a square is also a square.

**English Language Learners**

Define the midsegment of a trapezoid. Ask: How else could the midsegment have been defined? By joining the midpoints of the two bases. Why might this not be its definition? Resulting midsegment is not parallel to the legs.

learning style: verbal

**2. Teach**

**Guided Instruction**

**Connection to Algebra**

Help students understand that \( 2a \) is a valid coordinate by encouraging them to try the proof using other coordinates. They will see that the choice of variable is arbitrary.

**PowerPoint**

**Additional Examples**

1. Examine trapezoid \( TRAP \).
   Explain why you can assign the same \( y \)-coordinate to points \( R \) and \( A \). Since \( TP \parallel RA \) and \( TP \) is horizontal, \( RA \) is horizontal.

2. Use coordinate geometry to prove that the quadrilateral formed by connecting the midpoints of rhombus \( ABCD \) is a rectangle.

**Sample method:** Show that diagonals are congruent.

**Resources**

- Daily Notetaking Guide 6-7
- Daily Notetaking Guide 6-7—Adapted Instruction

**Closure**

How are the midsegments of trapezoids and triangles alike? How are they different? Both are parallel to bases; triangle midsegments are half the length of the third side, but trapezoid midsegments are the average length of both bases.
Problem Solving Hint
When you read large blocks of math text, cover all but a few lines to help you focus.

Developing Proof Complete the plan for each coordinate proof.

2. The diagonals of a parallelogram bisect each other (Theorem 6-3).

Given: Parallelogram $ABCD$

Prove: $AC$ bisects $BD$, and $BD$ bisects $AC$.

Plan: Place the parallelogram in the coordinate plane with a vertex at the origin and a side along the positive $x$-axis. Since midpoints will be involved, use multiples of $c$, $d$ to name coordinates. To show segments bisect each other, show the midpoints have the same $d$, $e$.

3. The diagonals of an isosceles trapezoid are congruent (Theorem 6-16).

Given: Trapezoid $EFGH$ with $FE \equiv GH$

Prove: $EG \equiv FH$

Plan: The trapezoid is isosceles, so place one base on the $x$-axis so that the $a$, $b$ bisects its bases. To show the diagonals are congruent, use the $b$, $e$ Formula.

4. The median to the hypotenuse of a right triangle is half the hypotenuse.

Given: $\triangle MNO$ is a right triangle with right $\angle MON$. $P$ is the midpoint of $MN$.

Prove: $OP = \frac{1}{2}MN$

Plan: Place the right triangle in the coordinate plane with the vertex of the $a$, $b$ at the origin and the $b$, $c$ along each axis. Since midpoints will be involved, use $c$, $d$ to name coordinates for points $d$, $e$ and $e$, $f$. Use the $f$, $g$ Formula to find the coordinates of $P$. To compare lengths, use the $g$, $h$ Formula.

5. The segments joining the midpoints of consecutive sides of an isosceles trapezoid form a rhombus. $a$-$g$. See margin.

Given: Trapezoid $TRAP$ with $TR \equiv TA$; $D$, $E$, $F$, and $G$ are midpoints of the indicated sides.

Prove: $DEFG$ is a rhombus.

Plan: The trapezoid is $a$, $b$ so place one base on the $b$, $c$ so that the $c$, $d$ bisects its bases. Use multiples of $2$ to name coordinates since $d$, $e$ will be involved. A rhombus is a parallelogram with four $e$, $f$. To show opposite sides are parallel, show that their $f$, $g$ are the same. To show sides are congruent, use $g$, $h$.

Developing Proof Follow the plans above to complete the coordinate proofs.

6. (Exercise 3) The diagonals of an isosceles trapezoid are congruent.

Proof: By the Distance Formula, $EG = a$, $b$ and $HF = b$, $b$. Therefore, $EG \equiv FH$ by the definition of congruence. $a$-$b$. See margin.

7. (Exercise 4) The median from the vertex of the right angle of a right triangle is half as long as the hypotenuse.

Proof: By the Distance Formula, $OP = a$, $b$ and $MN = b$, $b$. Therefore, $OP = \frac{1}{2}MN$. $a$-$b$. See margin.

Quick Check

1. $a$. $M(b, c), N(a + d, c)$; by starting with multiples of $2$ you eliminate fractions when using the midpoint formula.

b. $0, 0, 0$; they are =

c. $MN = d + a - b$,

$TP = 2a, RA = 2d - 2b$; so $RA + TP = 2d + 2a - 2b$ which is twice $MN$. So the midsegment is half the sum of the lengths of the bases.

d. The base along the $x$-axis allows us to calculate length by subtracting $x$-values.

2. $e$. isos.

b. $x$-axis

c. $y$-axis

d. midpoints

e. = sides

f. slopes

g. the Distance Formula
8. (Exercise 5) The segments joining the midpoints of consecutive sides of an isosceles trapezoid form a rhombus. **a–k. See margin.**

**Proof:** The midpoints have coordinates \( A(\frac{1}{2}, \frac{3}{2}), B(\frac{1}{2}, \frac{1}{2}), C(\frac{1}{2}, \frac{1}{2}), D(\frac{1}{2}, \frac{3}{2}) \) and \( E(\frac{1}{2}, \frac{1}{2}), F(\frac{1}{2}, \frac{1}{2}), G(\frac{1}{2}, \frac{1}{2}), H(\frac{1}{2}, \frac{1}{2}) \). By the Distance Formula, \( DE = b \), \( EF = c \), \( FG = d \), and \( GD = e \). The slope of \( DE = f \) and the slope of \( FG = g \). Thus, \( DEFG \) is a rhombus by the definition of rhombus.

9. **Developing Proof** Use coordinate geometry to prove that the diagonals of a rectangle bisect each other.

**Proof:** The midpoint of \( AC \) is \( a \). The midpoint of \( BD \) is \( b \). The midpoints are \( c \), so the diagonals bisect each other.

10. **Open-Ended** Give an example of a statement that you think is easier to prove with a coordinate geometry proof than with a paragraph, flow, or two-column proof. Explain your choice. **See back of book.**


**Given:** Kite \( DEFG \) with \( DE = EF \) and \( DG = GF \); \( K, L, M, \) and \( N \) are midpoints of the sides.

**Prove:** \( KLMN \) is a rectangle. **See back of book.**

State whether each type of conclusion shown here could be reached using coordinate methods. Give a reason for each answer. **12–23. See back of book.**

12. \( AB = CD \)
13. \( AB \parallel CD \)
14. \( AB \perp CD \)
15. \( \triangle ABC \) is isosceles.
16. \( \triangle ABC \) is equilateral.
17. \( \angle A = \angle B \)
18. \( \angle A = \angle B \)
19. \( AB + BC = AC \)
20. \( \angle A = \angle B \)
21. \( \triangle ABC \) is isosceles.
22. \( \triangle ABC \) is equilateral.
23. \( \triangle ABC \) is isosceles.
24. **Multiple Choice** \( DEFG \) is a rhombus. What is the slope of its diagonal \( DF \)?
   \[ A. \ 0 \quad B. \ \frac{c}{a} \quad C. \ \frac{d}{e} \quad D. \ \frac{a}{b} \quad E. \ undefined \]

A and \( B \) have coordinates \(-2 \) and \( 10 \) on a number line. Find the coordinates of the points that separate \( AB \) into the given number of congruent segments.

25. 4 \quad 26. 6 \quad 27. 10 \quad 28. 50 \quad 29. \( n \) \quad 25–29. See margin.

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30. (0, 7.5), (3, 10), (6, 12.5)  
31. (−1, 6.5), (1, 8.5), (3, 10), (5, 11.5), (7, 13)  
32. (−1.8, 6), (−0.6, 7), (0.6, 8), (1.8, 9), (3, 10), (4.2, 11), (5.4, 12), (6.6, 13), (7.8, 14)  

The endpoints of \( AB \) are \( A (−3, 5) \) and \( B (9, 15) \). Find the coordinates of the points that separate \( AB \) into the given number of congruent segments.

30. 4  
31. 6  
32. 10  
33. 50  
34. \( n \)  

33. (−2.76, 5.2), (−2.52, 5.4), (−2.28, 5.6), ..., (8.52, 14.6), (8.76, 14.8)  
34. (−3 + \( \frac{10}{n} \), 5 + \( \frac{10}{n} \)), (−3 + 2(\( \frac{1}{n} \)), 5 + 2(\( \frac{10}{n} \))), ..., (−3 + (n − 1)(\( \frac{10}{n} \)), 5 + (n − 1)(\( \frac{10}{n} \)))  

35. a. \( L(b, d), M(b + c, d), N(c, 0) \)  
b. \( AM: y = \frac{d}{b + c} x; \)  
   \( BN: y = \frac{2d}{b + c} (x − c); \)  
   \( CL: y = \frac{d}{b − 2c} (x − 2c); \)  
c. \( P = \frac{2(b + c)}{3}, \frac{2d}{3} \)  
d. Pt. \( P \) satisfies the eqs. for \( AM \) and \( CL \).  
e. \( AM = \sqrt{(b + c)^2 + 2d^2} \)  
   \( AP = \sqrt{\frac{2(b + c)^2}{2} + \left(\frac{2d}{3}\right)^2} = \frac{\sqrt{\frac{2}{3}}}{3} (b + c)^2 + d^2 = \frac{2}{3} AM \)  
   The other 2 distances are found similarly.

36. Complete the following steps to prove Theorem 5-9. You are given \( \triangle ABC \) with altitudes \( p, q, \) and \( r \). Show that \( p, q, \) and \( r \) intersect in a point (called the orthocenter of the triangle).  
a. The slope of \( BC \) is \( \frac{b}{c} \).  
   What is the slope of line \( p \)?  
b. Show that the equation of line \( p \) is \( y = \frac{b}{c} (x − a) \).  
c. What is the equation of line \( q \)?  
d. Show that lines \( p \) and \( q \) intersect at \( \left(0, \frac{ab}{c}\right) \).  
e. The slope of \( AC \) is \( \frac{a}{c} \).  
   What is the slope of line \( r \)?  
f. Show that the equation of line \( r \) is \( y = \frac{a}{c} (x − b) \).  
g. Show that lines \( r \) and \( q \) intersect at \( \left(0, \frac{ab}{c}\right) \).  
h. Give the coordinates of the orthocenter of \( \triangle ABC \).
Proof. 41. Write a coordinate proof of the theorem:
If the slopes of two lines have product \(-1\), the lines are perpendicular.

a. First, argue that neither line can be horizontal or vertical.
b. Then, tell why the lines must intersect. (Hint: Use indirect reasoning.)
c. Knowing that they do intersect, place the lines in the coordinate plane, choose a point on \(\ell_1\), find a related point on \(\ell_2\), and complete the proof. a–c. See back of book.

Test Prep

Multiple Choice
42. Two points on a line are \((-7, 10)\) and \((9, 2)\). Two points on a line parallel to that line are \((1, -3)\) and \((x, 4)\). What is the value of \(x\)?
A. \(-13\)  B. \(13\)  C. \(15\)  D. \(-15\)

43. Two points on a line are \((-4, 0)\) and \((8, 8)\). Two points on a line perpendicular to that line are \((8, -1)\) and \((6, y)\). What is the value of \(y\)?
G. 3  H. \(-\frac{7}{3}\)  J. \(-4\)

Short Response
44. The endpoints of a segment are \((7, 3)\) and \((a, b)\). The midpoint is \((3, 4)\).

a. What are the coordinates of the other endpoint? Show your work.
b. What is the length of the segment? Show your work. See back of book.

Extended Response
45. Given: \(\triangle ABC\), \(P, Q, R \) are the midpoints of \(\overline{AB}, \overline{AC}, \) and \(\overline{BC}\), respectively.
a. Place \(\triangle ABC\) in the coordinate plane by writing coordinates for \(A, B,\) and \(C\). a–c. See margin.
b. What are the coordinates of \(P, Q,\) and \(R\)?
c. Use coordinate geometry to prove \(\triangle APQ \cong \triangle RQP\) by SSS.

Mixed Review

Lesson 6-6
46. Rectangle \(LMNP\) at the right is centered at the origin. Give coordinates for point \(P\) without using any new variables. \((-a, b)\)

Lesson 5-4
47. If the sum of the angles of a polygon is not \(360^\circ\), then the polygon is not a quadrilateral.

48. If \(x \neq 51\), then \(2x \neq 102\).

49. If \(a \neq 5\), then \(a^2 \neq 25\).

50. If \(b < -4\), then \(b\) is not negative.

51. If \(c > 0\), then \(c\) is positive.

Lesson 4-4
49. Explain how you can use SSS, SAS, ASA, AAS with CPCTC to prove each statement true. 52–54. See margin.

52. \(\overline{AB} \cong \overline{CB}\)

53. \(\angle 1 \cong \angle 2\)

54. \(\angle K \cong \angle M\)

Online lesson quiz, PHSchool.com, Web Code: aua-0607

Lesson 6-7
40. Divide the quad. into 2 \(\triangle\). Find the centroid for each \(\triangle\) and connect them. Now divide the quad. into 2 other \(\triangle\) and follow the same steps. Where the two lines meet connecting the centroids of the 4 \(\triangle\) is the centroid of the quad.

45. [4] a-b. Sample:

\[ AP = \sqrt{(b - a)^2 + c^2} = RQ \]
\[ PQ = PQ \]
\[ AQ = a = RP \]
\[ \triangle APQ \cong \triangle RQP \] by SSS.

[3] minor computational error
[2] parts a and b correct
[1] one part correct

Test Prep

Resources
For additional practice with a variety of test item formats:
- Standardized Test Prep, p. 361
- Test-Taking Strategies, p. 356
- Test-Taking Strategies with Transparencies